Basis Vectors for a Radially Moving Point Source

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In Boyer-Lindquist co-ordinates, the Kerr metric is written

$$ds^2 = c^2\left(1 - \frac{2\mu r}{\rho^2}\right)dt^2 + \frac{4\mu a c r \sin^2\theta}{\rho^2}dtd\varphi - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 - \left(r^2 + a^2 + \frac{2\mu a^2 r \sin^2\theta}{\rho^2}\right)\sin^2\theta d\varphi^2$$

where,

$$\mu \equiv \frac{GM}{c^2}$$
$$\rho^2 \equiv r^2 + a^2\cos^2\theta$$
$$\Delta \equiv r^2 - 2\mu r + a^2$$

We wish to calculate the tetrad of basis vectors, $\left\{\mathbf{e}'_{(a)}\right\}$, describing the locally flat instantaneous rest frame of an observer (*i.e.* a point source) moving radially in the Kerr spacetime around a black hole.

The spacetime in the observer's rest frame reduces to Minkowski space, described by the metric

$$\mathbf{e}'_{(a)} \cdot \mathbf{e}'_{(b)} = \eta_{ab} \tag{1}$$

Where the Minkowski metric $[\eta_{ab}] = \mathrm{diag}(1, -1, -1, -1)$.

For a radially moving observer, travelling at velocity $\frac{dr}{dt} = V$, the 4-velocity can be written (in Boyer-Lindquist co-ordinates)

$$[u^\mu] = \left(u^t, u^r, 0, 0\right) = u^t\,(1, V, 0, 0) \tag{2}$$

Since the observer is at rest in its own rest frame, the spatial components of the 4-velocity must be zero in that frame, hence the observer's timelike basis vector is parallel to the 4-velocity. u^t is found by imposing the normalisation condition on the 4-velocity that $|\mathbf{u}| = c$ and working in natural units with $\mu = c = 1$ will result in a timelike unit vector.

$$\mathbf{u} \cdot \mathbf{u} = 1$$
$$g_{\mu\nu}u^\mu u^\nu = 1$$

$$g_{tt}\left(u^t\right)^2 + g_{rr}\left(u^r\right)^2 = 1$$
$$g_{tt}\left(u^t\right)^2 + g_{rr}V^2\left(u^t\right)^2 = 1$$

Hence,

$$u^t = \frac{1}{\sqrt{g_{tt} + g_{rr}V^2}} \tag{3}$$

And the timelike basis vector is given by

$$\boxed{\mathbf{e}'_{(t)} = \mathbf{u} = \frac{1}{\sqrt{g_{tt} + g_{rr}V^2}}\left(1, V, 0, 0\right)} \tag{4}$$

Given that the 4-velocity has no component in the θ direction, it can be seen from the metric that a basis vector can be found, that is orthogonal to those in the other co-ordinates, of the form

$$\mathbf{e}'_{(2)} = \left(0, 0, e^{\theta}_{(2)}, 0\right)$$

And the normalisation condition, $\mathbf{e}'_{(2)} \cdot \mathbf{e}'_{(2)}$ gives

$$\boxed{\mathbf{e}'_{(2)} = \left(0, 0, \frac{1}{\rho}, 0\right)} \tag{5}$$

In order to find a basis vector, $\mathbf{e}'_{(3)}$ corresponding to the radial direction, we note that the timelike basis vector has components in the t and r directions and hence try to find an orthogonal basis vector of the form

$$\mathbf{e}'_{(3)} = \left(e^t_{(3)}, e^r_{(3)}, 0, 0\right)$$

To be orthogonal to $\mathbf{e}'_{(t)}$,

$$\mathbf{e}'_{(t)} \cdot \mathbf{e}'_{(3)} = 0$$

$$g_{tt}e^t_{(t)}e^t_{(3)} + g_{rr}e^r_{(t)}e^r_{(3)} = 0$$

$$e^r_{(3)} = e^t_{(3)}\frac{g_{tt}}{-g_{rr}}\frac{e^t_{(t)}}{e^r_{(t)}}$$

Substituting $e^r_{(t)} = Ve^t_{(t)}$,

$$e^r_{(3)} = e^t_{(3)}\frac{g_{tt}}{-g_{rr}}\frac{1}{V}$$

And to normalise,

$$\mathbf{e}'_{(3)} \cdot \mathbf{e}'_{(3)} = -1$$

$$g_{tt}\left(e^t_{(3)}\right)^2 + g_{rr}\left(e^r_{(3)}\right)^2 = -1$$
$$g_{tt}\left(e^t_{(3)}\right)^2 + g_{rr}\left(e^t_{(3)}\right)^2\left(\frac{g_{tt}}{-g_{rr}}\right)\frac{1}{V^2} = -1$$

$$e^t_{(3)} = \sqrt{\frac{-g_{rr}}{g_{tt}}}\frac{V}{\sqrt{g_{tt} + g_{rr}V^2}} \tag{6}$$

And

$$e^r_{(3)} = \sqrt{\frac{g_{tt}}{-g_{rr}}}\frac{1}{\sqrt{g_{tt} + g_{rr}V^2}} \tag{7}$$

Hence,

$$\boxed{\mathbf{e}'_{(3)} = \left(\sqrt{\frac{-g_{rr}}{g_{tt}}}\frac{V}{\sqrt{g_{tt} + g_{rr}V^2}}, \sqrt{\frac{g_{tt}}{-g_{rr}}}\frac{1}{\sqrt{g_{tt} + g_{rr}V^2}}, 0, 0\right)}$$

Finally, to ensure orthogonality with the timelike and radial basis vectors, the fourth basis vector (corresponding to the azimuthal direction) could have components in the t, r and φ directions.

$$\mathbf{e}'_{(1)} = \left(e^t_{(1)}, e^r_{(1)}, 0, e^\varphi_{(1)}\right)$$

Orthogonality with the timelike basis vector gives

$$\mathbf{e}'_{(t)} \cdot \mathbf{e}'_{(1)} = 0$$

$$g_{tt}e^t_{(t)}e^t_{(1)} + g_{t\varphi}e^t_{(t)}e^\varphi_{(1)} + g_{rr}e^r_{(t)}e^r_{(1)} = 0$$
$$g_{tt}e^t_{(t)}e^t_{(1)} + g_{t\varphi}e^t_{(t)}e^\varphi_{(1)} + g_{rr}Ve^t_{(t)}e^r_{(1)} = 0$$

$$e^t_{(1)} = -\frac{\left(g_{t\varphi}e^\varphi_{(1)} + g_{rr}Ve^r_{(1)}\right)}{g_{tt}} \tag{8}$$

And orthogonality with $\mathbf{e}'_{(3)}$,

$$\mathbf{e}'_{(3)} \cdot \mathbf{e}'_{(1)} = 0$$

$$g_{tt}e^t_{(3)}e^t_{(1)} + g_{t\varphi}e^t_{(3)}e^\varphi_{(1)} + g_{rr}e^r_{(3)}e^r_{(1)} = 0$$

Equation 8 gives

$$-e^t_{(3)}\left(g_{t\varphi}e^{\varphi}_{(1)}+g_{rr}Ve^r_{(1)}\right)+g_{t\varphi}e^t_{(3)}e^{\varphi}_{(1)}+g_{rr}e^r_{(3)}e^r_{(1)}=0$$

$$g_{rr}e^r_{(1)}\left(e^r_{(3)}-Ve^t_{(3)}\right)=0$$

Which requires

$$e^r_{(1)}=0 \tag{9}$$

Hence, Equation 8 simplifies to

$$e^t_{(1)}=-\frac{g_{t\varphi}}{g_{tt}}e^{\varphi}_{(1)} \tag{10}$$

Lastly, normalising this basis vector,

$$\mathbf{e}'_{(1)}\cdot\mathbf{e}'_{(1)}=-1$$

$$g_{tt}\left(e^t_{(1)}\right)^2+2g_{t\varphi}e^t_{(1)}e^{\varphi}_{(1)}+g_{\varphi\varphi}\left(e^{\varphi}_{(1)}\right)^2=-1$$

Substituting for $e^t_{(1)}$,

$$\frac{g^2_{t\varphi}}{g_{tt}}\left(e^{\varphi}_{(1)}\right)^2-2\frac{g^2_{t\varphi}}{g_{tt}}\left(e^{\varphi}_{(1)}\right)^2+g_{\varphi\varphi}\left(e^{\varphi}_{(1)}\right)^2=-1$$

$$-\frac{g^2_{t\varphi}}{g_{tt}}\left(e^{\varphi}_{(1)}\right)^2+g_{\varphi\varphi}\left(e^{\varphi}_{(1)}\right)^2=-1$$

Giving

$$e^{\varphi}_{(1)}=\sqrt{\frac{g_{tt}}{g^2_{t\varphi}-g_{tt}g_{\varphi\varphi}}} \tag{11}$$

and

$$e^t_{(1)}=-\frac{g_{t\varphi}}{g_{tt}}\sqrt{\frac{g_{tt}}{g^2_{t\varphi}-g_{tt}g_{\varphi\varphi}}} \tag{12}$$

Hence, the tetrad of basis vectors describing the rest frame of a point source moving radially in the Kerr spacetime are given by

$$\boxed{\begin{aligned}
\mathbf{e}'_{(t)}&=\frac{1}{\sqrt{g_{tt}+g_{rr}V^2}}\left(1,V,0,0\right)\\
e^t_{(1)}&=\sqrt{\frac{g_{tt}}{g^2_{t\varphi}-g_{tt}g_{\varphi\varphi}}}\left(-\frac{g_{t\varphi}}{g_{tt}},0,0,1\right)\\
\mathbf{e}'_{(2)}&=\left(0,0,\frac{1}{\rho},0\right)\\
\mathbf{e}'_{(3)}&=\left(\sqrt{\frac{-g_{rr}}{g_{tt}}}\frac{V}{\sqrt{g_{tt}+g_{rr}V^2}},\sqrt{\frac{g_{tt}}{-g_{rr}}}\frac{1}{\sqrt{g_{tt}+g_{rr}V^2}},0,0\right)
\end{aligned}}$$