Geodesic Equations in Kerr Spacetime

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In Boyer-Lindquist co-ordinates, the Kerr metric is written

$$\begin{aligned} ds^2 = c^2\left(1 - \frac{2\mu r}{\rho^2}\right)dt^2 \quad &+ \quad \frac{4\mu acr\sin^2\theta}{\rho^2}dtd\varphi - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 \\ &- \quad \left(r^2 + a^2 + \frac{2\mu a^2 r\sin^2\theta}{\rho^2}\right)\sin^2\theta d\varphi^2 \end{aligned}$$

where,

$$\begin{aligned} \mu &\equiv \frac{GM}{c^2} \\ \rho^2 &\equiv r^2 + a^2\cos^2\theta \\ \Delta &\equiv r^2 - 2\mu r + a^2 \end{aligned}$$

The Lagrangian is given by

$$\mathcal{L} = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$$

which for the Kerr metric gives:

$$\begin{aligned} \mathcal{L} = c^2\left(1 - \frac{2\mu r}{\rho^2}\right)\dot{t}^2 \quad &+ \quad \frac{4\mu acr\sin^2\theta}{\rho^2}\dot{t}\dot{\varphi} - \frac{\rho^2}{\Delta}\dot{r}^2 - \rho^2\dot{\theta}^2 \\ &- \quad \left(r^2 + a^2 + \frac{2\mu a^2 r\sin^2\theta}{\rho^2}\right)\sin^2\theta\dot{\varphi}^2 \end{aligned}$$

The geodesic equations are then given by the Euler-Lagrange equation:

$$\frac{d}{d\sigma}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}\right) - \frac{\partial\mathcal{L}}{\partial x^\mu} = 0$$

For the t co-ordinate, this gives

$$\frac{d}{d\sigma}\left[c^2\left(1 - \frac{2\mu r}{\rho^2}\right)\dot{t} + \frac{2\mu acr\sin^2\theta}{\rho^2}\dot{\varphi}\right] = 0$$

Thus, this is a constant of the motion.

$$c^2\left(1 - \frac{2\mu r}{\rho^2}\right)\dot{t} + \frac{2\mu acr\sin^2\theta}{\rho^2}\dot{\varphi} \equiv kc^2 \tag{1}$$

Likewise, the Euler-Lagrange equation gives for φ,

$$\frac{d}{d\sigma}\left[\frac{2\mu acr\sin^2\theta}{\rho^2}\dot{t}-\left(r^2+a^2+\frac{2\mu a^2 r\sin^2\theta}{\rho^2}\right)\sin^2\theta\dot{\varphi}\right]=0$$

Which is also a constant

$$\frac{2\mu acr\sin^2\theta}{\rho^2}\dot{t}-\left(r^2+a^2+\frac{2\mu a^2 r\sin^2\theta}{\rho^2}\right)\sin^2\theta\dot{\varphi}\equiv -h \tag{2}$$

Eliminating between Equations 1 and 2 yields the geodesic equation for $\dot{\varphi}$.

$$\dot{\varphi}=\frac{\frac{2\mu acr}{\rho^2}k+\left(1-\frac{2\mu r}{\rho^2}\right)\frac{h}{\sin^2\theta}}{r^2+a^2-\frac{2\mu r^3}{\rho^2}-\frac{2\mu a^2 r\cos^2\theta}{\rho^2}} \tag{3}$$

Which, after multiplying the fraction through by $\rho^2\sin^2\theta$ and some further manipulation, can be re-written

$$\dot{\varphi}=\frac{2\mu acrk\sin^2\theta+(r^2+a^2\cos^2\theta-2\mu r)h}{(r^2+a^2)(r^2+a^2\cos^2\theta-2\mu r)\sin^2\theta+2\mu a^2 r\sin^4\theta} \tag{4}$$

The geodesic equation for $\dot{t}$ is given by substituting (3) back into Equation 2:

$$\dot{t}=\frac{\left(r^2+a^2+\frac{2\mu a^2 r\sin^2\theta}{\rho^2}\right)k-\frac{2\mu ar}{c\rho^2}h}{r^2+a^2-\frac{2\mu r^3}{\rho^2}-\frac{2\mu a^2 r\cos^2\theta}{\rho^2}} \tag{5}$$

After multiplying the fraction through by ρ^2, some further manipulation gives

$$\dot{t}=\frac{\left[(r^2+a^2\cos^2\theta)(r^2+a^2)+2\mu a^2 r\sin^2\theta\right]k-\frac{2\mu ar}{c}h}{r^2\left(1+\frac{a^2\cos^2\theta}{r^2}-\frac{2\mu}{r}\right)(r^2+a^2)+2\mu a^2 r\sin^2\theta} \tag{6}$$

It is possible to obtain equivalent second order expressions for $\ddot{r}$ and $\ddot{\theta}$, however it is often more convenient to obtain first order expressions using first integrals.

A first integral of the geodesic equations is available from the constant length of the 4-velocity

$$g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu=\epsilon^2$$

Where $\epsilon^2=c^2$ for massive particles and $\epsilon^2=0$ for photons (null geodesics).

Furthermore, for the Kerr metric it can be shown that (see, e.g. 'The Mathematical Theory of Black Holes', S. Chandrasekhar, 1983)

$$\left[a\sin\theta c\dot{t}-(r^2+a^2)\sin\theta\dot{\varphi}\right]^2+\rho^4\dot{\theta}^2+\epsilon a^2\cos^2\theta\equiv K$$

is conserved along an affinely-parameterised geodesic.

Using (1) and (2),

$$a\sin\theta\dot{t}-(r^2+a^2)\sin\theta\dot{\varphi}=kca\sin\theta-h\text{cosec }\theta$$

Thus,

$$\rho^4\dot{\theta}^2 = K - \epsilon a^2\cos^2\theta - (kca\sin\theta - h\text{cosec }\theta)^2$$

This can be rewritten

$$\rho^4\dot{\theta}^2 = K - (h-kca)^2 - (\epsilon - k^2c^2)a^2\cos^2\theta - h^2\cot^2\theta$$

For null geodesics (*i.e.* photons), $\epsilon = 0$.

$$\rho^4\dot{\theta}^2 = K - (h-kca)^2 + k^2c^2a^2\cos^2\theta - h^2\cot^2\theta$$

And redefining the constant, $Q = K - (h-kca)^2$,

$$\dot{\theta}^2 = \frac{Q + (kca\cos\theta - h\cot\theta)(kca\cos\theta + h\cot\theta)}{\rho^4} \tag{7}$$

Finally, the geodesic equation in $\dot{r}$ is readily obtained from the length of the 4-velocity.

$$g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \epsilon^2$$

Putting in the components of the metric,

$$\begin{aligned} c^2\left(1-\frac{2\mu r}{\rho^2}\right)\dot{t}^2 \quad &+ \quad \frac{4\mu acr\sin^2\theta}{\rho^2}\dot{t}\dot{\varphi} - \frac{\rho^2}{\Delta}\dot{r}^2 - \rho^2\dot{\theta}^2 \\ &- \quad \left(r^2 + a^2 + \frac{2\mu a^2 r\sin^2\theta}{\rho^2}\right)\sin^2\theta\dot{\varphi}^2 = \epsilon^2 \end{aligned}$$

Substituting using (2) gives

$$\frac{\rho^2}{\Delta} = c^2\left(1-\frac{2\mu r}{\rho^2}\right)\dot{t}^2 + \frac{2\mu acr\sin^2\theta}{\rho^2}\dot{t}\dot{\varphi} - \rho^2\dot{\theta}^2 - h\dot{\varphi} - \epsilon^2$$

And finally using (1),

$$\dot{r}^2 = \frac{\Delta}{\rho^2}\left[kc^2\dot{t} - h\dot{\varphi} - \rho^2\dot{\theta}^2 - \epsilon^2\right] \tag{8}$$

Which is readily evaluated once $\dot{t}, \dot{\theta}$ and $\dot{\varphi}$ have been found.

Summary of Geodesic Equations

$$\begin{aligned} \dot{t} &= \frac{\left[(r^2+a^2\cos^2\theta)(r^2+a^2) + 2\mu a^2 r\sin^2\theta\right]k - \frac{2\mu ar}{c}h}{r^2\left(1+\frac{a^2\cos^2\theta}{r^2} - \frac{2\mu}{r}\right)(r^2+a^2) + 2\mu a^2 r\sin^2\theta} \\ \dot{\varphi} &= \frac{2\mu acrk\sin^2\theta + (r^2+a^2\cos^2\theta - 2\mu r)h}{(r^2+a^2)(r^2+a^2\cos^2\theta-2\mu r)\sin^2\theta + 2\mu a^2 r\sin^4\theta} \\ \dot{\theta}^2 &= \frac{Q + (kca\cos\theta - h\cot\theta)(kca\cos\theta + h\cot\theta)}{\rho^4} \\ \dot{r}^2 &= \frac{\Delta}{\rho^2}\left[kc^2\dot{t} - h\dot{\varphi} - \rho^2\dot{\theta}^2 - \epsilon^2\right] \end{aligned}$$