# Numerical Calculation of Basis Vectors

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In order to conduct ray tracing calculations originating from a source in the Kerr spacetime around a black hole, it is often necessary to construct the tetrad of basis vectors, $\left\{\mathbf{e}'_{(a)}\right\}$, describing the locally flat instantaneous rest frame of that source (an observer). Given these basis vectors, rays can be constructed, emitted isotropically in the source's own rest frame (*i.e.* at equal intervals in solid angle) and then transformed to Boyer-Lindquist co-ordinates such that the propagation of the rays around the black hole can be computed.

While analytic solutions can be found for specific cases, for instance azimuthal or radial motion of the source, it can be desirable to compute the basis vectors numerically for arbitrary motion of the source, described by its 4-velocity $\mathbf{u}$.

The spacetime in the observer's rest frame reduces to Minkowski space, described by the metric

$$\mathbf{e}'_{(a)} \cdot \mathbf{e}'_{(b)} = \eta_{ab} \tag{1}$$

Where the Minkowski metric $[\eta_{ab}] = \mathrm{diag}\,(1, -1, -1, -1)$.

Since the observer is at rest in its own rest frame, the spatial components of the 4-velocity must be zero in that frame, hence the observer's timelike basis vector is parallel to the 4-velocity. $u^t$ is found by imposing the normalisation condition on the 4-velocity that $|\mathbf{u}| = c$ and working in natural units with $\mu = c = 1$ will result in a timelike unit vector.

$$\hat{\mathbf{e}}_0 = \mathbf{u} \tag{2}$$

The three spacelike basis vectors, which are orthogonal to the timelike unit vector and to each other, can be computed using the Gram-Schmidt orthogonalisation procedure.

We begin with an initial guess of the basis vectors, $\{\mathbf{v}_a\}$. These are initially taken to be in the $r$, $\theta$ and $\varphi$ directions in Boyer-Lindquist co-ordinates. Note that in order to prioritise obtaining a basis vector corresponding to the radial direction to align with the polar axis of the source frame, $\mathbf{v}_1$ is taken to be in the radial direction. While there are an infinite number of possible tetrads (corresponding to rotations of the co-ordinate axes of the source frame), this will result in a vector as close to radial as is possible to be orthogonal with the 4-velocity and then will construct the other two basis vectors to be orthogonal to this).

$$\mathbf{v}_1 = (0, 1, 0, 0)$$
$$\mathbf{v}_2 = (0, 0, 1, 0)$$
$$\mathbf{v}_3 = (0, 0, 0, 1)$$

Having set the initial basis vector to be the 4-velocity, the Gram-Schmidt procedure then creates the orthogonal tetrad by taking each of these starting vectors and subtracting the projections of this vector in the directions of the previously found basis vectors, hence ensuring that each vector has no component in the direction of the others.

For $i = 1, 2, 3$

$$\mathbf{e}_i = \mathbf{v}_i - \sum_{j=0}^{i-1} \frac{\mathbf{v}_i \cdot \mathbf{e}_j}{\mathbf{e}_j \cdot \mathbf{e}_j} \mathbf{e}_j \tag{3}$$

Where the scalar product is computed using the metric $\mathbf{u} \cdot \mathbf{v} = g_{\mu\nu} u^\mu v^\nu$.

Finally, the resulting set of orthogonal vectors is normalised to give the tetrad of unit basis vectors.

$$\hat{\mathbf{e}}_a = \frac{\mathbf{e}_a}{|\mathbf{e}_a|} = \frac{\mathbf{e}_a}{\sqrt{|\mathbf{e}_a \cdot \mathbf{e}_a|}} \tag{4}$$